Cypress College Math Review:

Second Order Differential Equations Special Types

In general a 2nd order differential equation can be written as $F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) = 0$

There are two special types of 2nd order differential equations where a substitution reduces them to a 1st order differential equation that we can solve.

Type 1 (no x): $F\left(t, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) = 0$ **The dependent variable is missing.**

Substitute: $v = \frac{dx}{dt}$ then $\frac{d^2x}{dt^2} = \frac{dv}{dt} = v'$

Example) $t^2\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 = 2t\frac{dx}{dt}$

Example) $2t\frac{d^2x}{dt^2}=\left(\frac{dx}{dt}\right)^2-1$

Type 2 (no t): $F\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) = 0$ **The independent variable is missing.**

Substitute: $v = \frac{dx}{dt}$ then $\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$

Example) $\left(x^2 + 1\right)\frac{d^2x}{dt^2} = 2x\left(\frac{dx}{dt}\right)^2$

Example) $y'' + y\left(y'\right)^3 = 0$

Extra Practice – Try these on your own, then check with the answers below.

1. $x\dfrac{d^2x}{dt^2}=\left(\dfrac{dx}{dt}\right)^2$
2. $t^2y"+2ty'-1=0,\ t>0$
3. $y^2y"-\left(y'\right)^3=0$

Answers

1. $x=C_2e^{C_1t}$, x = C is also a solution
2. $y=\ln t-\dfrac{C_1}{t}+C_2$
3. $\ln\left|y\right|-C_1y=t+C_2$, y = C is also a solution